Geometric phases in quantum mechanics

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We give an introductory overview of how geometric ideas naturally emerge from a study of geometric phases in nonrelativistic quantum mechanics, focusing primarily on Berry's phase. We expand on the notions of $U(1)$ gauge symmetry and parallel transport, eventually characterizing our simple system as a complex hermitian line bundle. We give a few examples, as well as comment on some generalizations which go beyond adiabaticity and display the fundamental importance of holonomy on quantum evolution.

I. INTRODUCTION AND OUTLINE

Geometric phases tend to represent subtle manifestations of nontrivial geometric ideas. This is clearly seen in the example of Berry's phase, which will be our main focus. Starting from quite simple nonrelativistic quantum mechanics in the context of adiabatic evolution, one quickly finds themselves talking about concepts such as parallel transport, principal bundles etc. That this phase is a subtlety with a lot to say is also confirmed by history. The appearance of a geometric phase was indeed first noticed and dismissed as irrelevant in the Born-Oppenheimer approximation, until Berry [1] rediscovered it and noted it was actually observable. Immediately after Berry's discovery of his eponymous phase, Simon[1] has shown how to interpret this geometrically [2]. We will try to focus on elucidating its geometric significance.

This work is divided into three parts. In the first part we introduce Berry's phase. In the second part we try to delve into the geometry at work. In the third and final part we give examples and some generalizations. We also give some extra relations in the appendix.

We noted that geometric phases pack a conceptual punch. One does need a lot of machinery as well as inutition to understand them. We will try to keep this discussion as self-contained as possible. If we have failed, we apologise and refer the reader to two ambitious classics, [6] and [7].

II. PRELIMINARIES

A. Adiabaticity

The usual derivation of geometric phases follows directly from the Schrödinger equation. The central assumption in this approach is that of *adiabaticity*, for which we need a suitable definition. Consider therefore a time-dependent Hamiltonian with a discrete and nondegenerate spectrum[2]. Since the eigensystem will depend on time, we will pick a set of instantaneous eigenvectors $\{|n(t)\rangle\}_{n\in S}$, where $S \subseteq \mathbb{N}$ is the index set of (for simplicity) a finite number of eigenvectors. These are required to satisfy

$$H(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle \tag{1}$$

and also need to be orthogonal, ie

$$\langle n(t)|m(t)\rangle = \delta_{nm}, \forall t \tag{2}$$

where we have not specified the interval t belongs to as this is immaterial to our discussion.

Adiabaticity could in an intuitive way be called "static dynamics". We require a time evolution slow enough not to bring about sudden changes to our system, akin to pulling upwards a rock tied to a string - yank it and the string will undoubtedly snap. In effect, we are considering any characteristic times in our system to be much shorter than characteristic times associated to the adiabatic evolution. We can capture this intuition by requiring near-simultaneous states remain orthogonal:

$$\langle n(t)|m(t+\delta t)\rangle \overset{!}{=} 0 \tag{3}$$

$$\Rightarrow \langle n(t)|\,\partial_t\,|m(t)\rangle = 0 \tag{4}$$

where we take $m \neq n$. Note that we should not be taking these equals signs too literally, since adiabaticity is after all a certain limit of a process. It might be better to write "$\to$" etc. A way to quantify this is by applying ∂_t to the eigenvector equation itself

$$(\partial_t H(t))\,|n(t)\rangle + H(t)\partial_t\,|n(t)\rangle = (\partial_t E_n(t))\,|n(t)\rangle + E_n(t)\partial_t\,|n(t)\rangle$$

Acting on this by $\langle m(t)|$, and no longer writing time dependance to reduce clutter gives us

$$\langle m|\,\partial_t\,|n\rangle = -\frac{\langle m|\,\partial_t H\,|n\rangle}{E_m - E_n} \tag{5}$$

We may now define the absolute value of the RHS to be $1/T_{mn}$. We see that adiabaticity amounts to $T_{mn} \to \infty$, which we may interpret as some time of transition between the two states. However we interpret it, it is clearly a time scale associated to adiabatic evolution, assumed large, and not associated to any characteristic time scales of our system.

[1] Curiously, Simon's work was published before Berry's.
[2] NB: this assumption may be relaxed to Hamiltonians satisfying certain gap conditions. This would take us off track, so we refer the interested reader to [5].

B. The usual derivation

Now that we have cleared that up we will consider the evolution of the system in some state $|\psi(t)\rangle$ which we expand in our eigenbasis and write

$$|\psi(t)\rangle = \sum_{m\in S} c_m(t) e^{-\frac{i}{\hbar}\int_0^t dt' E_m(t')} |m(t)\rangle \tag{6}$$

We can act on this by ∂_t, reexpress it using the Schrödinger equation and then act by with $\langle n(t)|$. Adiabaticity however kills all terms except the $m = n$ one, and we obtain

$$\dot{c}_m = -c_m \langle m| \partial_t |m\rangle . \tag{7}$$

Now consider a system initially prepared in the n-th eigenstate. This means we have $|\phi(0)\rangle = |n(0)\rangle$, which means that $c_m(0) = \delta_{mn}$, giving us, finally,

$$|\psi(t)\rangle = c_n(t) e^{-\frac{i}{\hbar}\int_0^t dt' E_n(t')} |n(t)\rangle \tag{8}$$

with an additional phase

$$c_n(t) = \exp\left\{ - \int_0^t dt' \langle n| \partial_{t'} |n\rangle \right\} \tag{9}$$

which we will write as

$$c_n(t) = \exp\left\{ i \int_0^t dt' \langle n| \partial_{t'} |n\rangle \right\} \tag{10}$$

since $\langle n| \partial_{t'} |n\rangle$ is purely imaginary[3]. We see that adiabaticity entails that a system initally prepared in an eigenstate will only change up to phase.

C. Introducing a parameter space

In reality, we evolve our system by manipulating a given set of independent parameters. For instance, we may be dealing with an electron moving in a plane perpendicular to a solenoid, so that the parameters are the electron's x,y positions. In this setup we get the Aharonov-Bohm effect, which will be discussed later. We will therefore assume our system is located at a certain point $X = (X_1, \ldots, X_d)$ in an d-dimensional parameter space M, assumed to be a manifold.

We have minor modifications to make. First of all, we will write $|n(t)\rangle$ as $|n(X)\rangle$, although we will often omit this altogether. Next, we only need modify the additional phase we obtained in the last section:

$$\begin{aligned} &\mathrm{Im} \int_0^t dt' \langle n| \partial_{t'} |n\rangle && (11)\\ =&\mathrm{Im} \int_0^t dt' \langle n(X)| \boldsymbol{\nabla}_X |n(X)\rangle \cdot \frac{\partial \mathbf{X}}{\partial t'} && (12)\\ =&\mathrm{Im} \int_C \mathrm{d}\mathbf{X} \cdot \langle n(X)| \boldsymbol{\nabla}_X |n(X)\rangle && (13)\\ =&\mathrm{Im} \int_C \langle n(X)|\mathrm{d}n(X)\rangle && (14) \end{aligned}$$

where we the last line defines a 1-form $\langle n|\partial_k n\rangle \, \mathrm{d}X^k$.

Now consider a system adiabatically undergoing a loop in the parameter space. In this case we will call this phase *Berry's phase* and label it γ_C. If the curve is such that Stokes' theorem is applicable, we have

$$\gamma_C = \mathrm{Im} \oint \langle n|\mathrm{d}n\rangle = \mathrm{Im} \iint_\Sigma \langle \mathrm{d}n| \wedge |\mathrm{d}n\rangle \tag{15}$$

where $C = \partial\Sigma$. The 2-form is called Berry's curvature, and some further properties will be given in the appendix.

Now, the reader may have been rolling her eyes ever since it got clear to her that we were to obtain a phase. She interjects: *It is basic quantum mechanics that states are defined up to a phase! We may as well have taken* $|\tilde{n}(t)\rangle = e^{i\lambda_n(t)} |n(t)\rangle$, $\forall n \in S$ *as our basis, so we can simply gauge away the phase.*

True, this defines a special basis, in the so-called Born-Fock gauge. Following the reader, we see that by choosing exactly $|\tilde{n}(t)\rangle = e^{i\lambda_n(t)} |n(t)\rangle$, where $\partial_t \lambda_n = i \langle n|\dot{n}\rangle$ (and *not* $i \langle \tilde{n}|\dot{\tilde{n}}\rangle$, since this is now zero!) we eliminate the geometric phase, and we get

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t dt' E_n(t')} |\tilde{n}(t)\rangle \tag{16}$$

Now let's follow a closed path $C \in M$ with base point X_0, and consider a locally well-defined basis $|n(X)\rangle$, single valued at X_0. After going following the loop, we have

$$|\tilde{n}_f\rangle = e^{i(\gamma_C + \gamma_d)} |n(X_0)\rangle = e^{i(\gamma_C+\gamma_d)} |\tilde{n}_i\rangle \tag{17}$$

so there is a phase difference between the initial and final states, given by both Berry's phase and a dynamical phase. We have tried to hide the phase differences by incessantly $U(1)$-rotating our vector, and the result is a nonzero, observable rotation. It could happen that this geometric phase vanishes for every loop, but in general $|\tilde{n}\rangle$ will only be locally defined since it's a certian vector field on a manifold.

A way to measure this would be to prepare two particles in the same eigenstate, transport only one and then measure interference. It turns out that it's easier to prepare a superposition of different eigenstates, so that

$$|\psi_i\rangle = \sum_n a_n |n\rangle \tag{18}$$

$$|\psi_f\rangle = \sum_n a_n |n\rangle \, e^{i(\gamma_{C,n} + \gamma_{d,n})} \tag{19}$$

[3] Since $0 = \partial_t \langle n|n\rangle = \langle \dot{n}|n\rangle + \langle n|\dot{n}\rangle = 2\mathrm{Re} \langle n|\dot{n}\rangle$

and then measure an observable which doesn't commute with the Hamiltonian to get interference. Note that we can get rid of the dynamical phase altogether by replacing the Hamiltonian with $H - E_n(t)\mathbb{1}$.

III. GEOMETRY

A. U(1) gauge invariance

In this and the following sections we will attempt to give a geometric interpretation of Berry's phase. First of all, we note that it's gauge invariant, in the sense that $|n(x)\rangle \mapsto e^{i\beta(X)}|n(x)\rangle$, with let's say $\beta \in C^\infty(M)$. It is very simple to see that γ_C is invariant since we have

$$i\langle n|\mathrm{d}n\rangle \mapsto i\langle n|\mathrm{d}n\rangle - \mathrm{d}\beta \tag{20}$$

But this is a $U(1)$ gauge transformation and this is exactly how such gauge connections transform. We therefore define the Simon connection,

$$\mathcal{A}_n = i\langle n|\mathrm{d}n\rangle = -\mathrm{Im}\langle n|\mathrm{d}n\rangle \tag{21}$$

We already saw that it is purely imaginary, and that

$$\gamma_C = \int_C A_n = i\int_\Sigma \langle \mathrm{d}n| \wedge |\mathrm{d}n\rangle \equiv \int_\Sigma F \tag{22}$$

defines a curvature - Berry's curvature.

Now we see that in order to be gauge invariant the (single particle) Hamiltonian needs to have a kinetic part given by

$$H_{kin} = \frac{1}{2m}\left(p - e\mathcal{A}_n\right)^2 \tag{23}$$

This is nothing more than the Born-Oppenheimer approximation, it turns out [3]. Here p takes care of the fast degrees of freedom, and the connection represents the "frozen", slow ones. However, we don't even need the Hamiltonian to be gauge invariant. The gauge transformation will give a family of Hamiltonians $H[\beta(X)]$ with the common property of having one representative of these "gauged" kets as an eigenket. Berry's phase is thus connected purely to holonomy and will remain unchanged. The dynamical one will not.

B. Parallel transport

We will take a detour into parallel transport on manifolds. We repeat Berry's illustration [3] which will hopefully elucidate many things. Consider therefore parallel transport along a curve $C \subseteq S^2$. The curve will be swept out by $\mathbf{r}(t) \in S^2$ and we will be transporting a (unit) vector $\mathbf{e}(t) \in TS^2$ along it. By viewing both S^2 and TS^2 as being embedded in $\mathbb{R}^3$, we can form the usual dot product between vectors in both these spaces, without needing any extra constructions. First of all, we clearly have

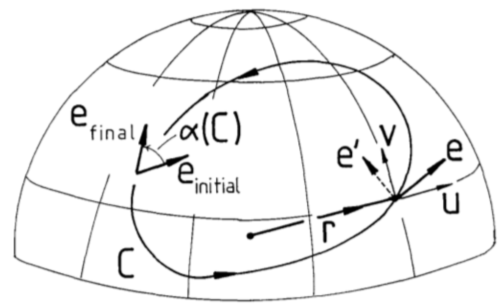

Figure 1: Parallel transport on S^2, borrowed from [3].

$$\mathbf{r}(t)\cdot\mathbf{e}(t) = 0 \tag{24}$$

at all times. We also demand the length of the vector to be constant, which implies $\dot{\mathbf{e}}(t)\cdot\mathbf{e}(t) = 0$. This means we can write

$$\dot{\mathbf{e}} = \mathbf{\Omega}\times\mathbf{e} \tag{25}$$

Now to determine $\mathbf{\Omega}$, we note that $\{\mathbf{r}, \dot{\mathbf{r}}, \mathbf{r}\times\dot{\mathbf{r}}\}$ form an orthogonal basis. Now let's find the change in $\mathbf{r}$:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{e}) = \dot{\mathbf{r}}\cdot\mathbf{e} + \mathbf{r}\cdot(\mathbf{\Omega}\times\mathbf{e}) \\ &= (\dot{\mathbf{r}} - \mathbf{\Omega}\times\mathbf{r})\cdot\mathbf{e} \end{aligned} \tag{26}$$

So, $\mathbf{r}$ has the same evolution, except it's pointing radially out of the sphere at the start. Now we can put

$$\mathbf{\Omega} = c_1\dot{\mathbf{r}} + c_2\mathbf{r}\times\dot{\mathbf{r}} \tag{27}$$

We have excluded the component of the form $c_3\mathbf{r}$, because that would imply $\mathbf{e}$ rotates around $\mathbf{r}$. Plugging this back and expanding the triple vector product gives

$$0 = (\dot{\mathbf{r}} - c_2\dot{\mathbf{r}} + c_1\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{e} = 0 \tag{28}$$

which implies $c_1 = 0$ and $c_2 = 1$. We can now write

$$\dot{\mathbf{e}} = -(\mathbf{e}\cdot\dot{\mathbf{r}})\,\mathbf{r} \tag{29}$$

After a closed circuit, the vector $\mathbf{e}$ will fail to be parallel to how it was at start, gaining instead an angle $\alpha(C)$, despite never having been actually rotated. Now consider transporting an orthonormal frame $(\mathbf{e}, \mathbf{e}')$. We may clearly take $\mathbf{e}' = \mathbf{r}\times\mathbf{e}$, and we can verify that the entire frame will be rotated by the same amount $\alpha(C)$. We would like to measure this using a local basis. Recall that S^2 is not parallelizable ("hairy ball" theorem!) so there are no globally defined nonvanishing vector fields. Still, if we agree to avoid let's say the poles, we the two vectors

$$\mathbf{u} = (-\sin\phi, \cos\phi, 0) = \hat{\phi} \tag{30}$$

$$\mathbf{v} = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta) = -\hat{\theta} \tag{31}$$

which provide us with a single valued basis. Along C, these two bases are connected via a t-dependent $SO(2)$ transformation, but since $SO(2) \cong U(1)$ and we like $U(1)$ better, we should complexify them.

Consider two therefore vectors spanned by these bases.

$$\mathbf{n}_e = \frac{1}{\sqrt{2}} \left(\mathbf{e} + i\mathbf{e}'\right) \tag{32}$$

$$\mathbf{n} = \frac{1}{\sqrt{2}} \left(\mathbf{u} + i\mathbf{v}\right) \tag{33}$$

and note that the constancy of the vector's lenght automatically gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{n}_e^* \cdot \mathbf{n}_e\right) = 2\mathrm{Re}\{\mathbf{n}_e^* \cdot \dot{\mathbf{n}}_e\} \tag{34}$$

Since we have likewise $\dot{\mathbf{n}}_{\mathbf{e}} = -(\mathbf{n}_{\mathbf{e}} \cdot \dot{\mathbf{r}})\mathbf{r}$, we compute

$$\mathbf{n}_e^* \cdot \dot{\mathbf{n}}_e = -\left(\mathbf{n}_e^* \cdot \mathbf{r}\right)\left(\mathbf{n}_e \cdot \dot{\mathbf{r}}\right) = 0 \tag{35}$$

since $\mathbf{n}_e$ and its complex conjugate are orthogonal to $\mathbf{r}$. Since the vector is already normalized, this leaves us with only the condition

$$\mathrm{Im}\left(\mathbf{n}_e^* \cdot \dot{\mathbf{n}}_e\right) = 0 \quad \Rightarrow \quad \mathrm{Im}\left(\mathbf{n}_e^* \cdot \mathrm{d}\mathbf{n}_e\right) = 0 \tag{36}$$

where we promoted the infinitesimal change during dt to the exterior derivative of the section, which we can view simply as a vector-valued 1-form - more on that later.

We can finally find the angle $\alpha(C)$. Using a $U(1)$ transformation we relate

$$\mathbf{n}_e(t) = e^{i\alpha(t)}\mathbf{n}(t), \quad \alpha(t) \in \mathbb{R} \tag{37}$$

Plugging it back into the above condition gives

$$d\alpha = \mathrm{Im}\left(\mathbf{n}^* \cdot \mathrm{d}\mathbf{n}\right) \tag{38}$$

And therefore we get a very familiar phase back:

$$\alpha_C = \oint d\alpha = \mathrm{Im} \oint \left(\mathbf{n}^* \cdot \mathrm{d}\mathbf{n}\right) \tag{39}$$

$$= \mathrm{Im} \iint \left(\mathrm{d}\mathbf{n}^* \wedge \cdot \mathrm{d}\mathbf{n}\right) \tag{40}$$

Computing this area element in terms of

$$\mathbf{n} = \frac{1}{\sqrt{2}} \left(\hat{\phi} - i\hat{\theta}\right) \tag{41}$$

gives us $\mathrm{d}\theta\mathrm{d}\phi \sin\phi$, so that $\alpha(C) = \Omega(C)$, the solid angle subtended by C viewed from the origin.

We can clearly see the analogy with our previous considerations. We showed that for a special vector $|\tilde{n}(X)\rangle$ whose connection (21) vanishes, ie

$$\langle \tilde{n}(X) | \mathrm{d}\tilde{n}(X) \rangle = 0 \tag{42}$$

we obtain Berry's phase after going along a closed path. We also had to connect this to a single-valued frame, also by a $U(1)$ transformation. However, we didn't start with this condition, we saw it must hold if we wanted to gauge the phase away.

Note that adiabaticity made us demand orthogonality hold infinitesimally

$$\langle n(t) | m(t + \delta t) \rangle = 0 \tag{43}$$

$$\Rightarrow \langle n(t) | \, \partial_t \, | m(t) \rangle = 0 \tag{44}$$

however we didn't ask for this to be true when $|m\rangle = |n\rangle$. What would that imply geometrically is clear. We move the vector around so that it remains parallel to itself at nearby points and this is nothing more than the definition of parallel transport.

C. Line bundles

We now have a clear geometric picture of Berry's phase and we have freed ourselves from the schackles of Hamiltonians. What remains are $U(1)$ gauge invariant vectors which live on M. Since we associate a complex line $L_X \cong U(1)$ to each point $X \in M$, whose elements are identified under a $U(1)$ gauge transformation and since we assume L_X vary smoothly with X, we are dealing with a hermitian line bundle $P(U(1), M)$ over M. This is exactly the definition of such an object. We can regard $|n(X)\rangle$ as basis vectors of the lines L_X and therefore as sections. We will call the total space E. The canonical projection amounts to $\pi(e^{i\beta(X)} |n(X)\rangle) = X$, and a local section is a complex vector field $|\mathbf{v}\rangle = v(X) |n(X)\rangle$

We define covariant differentiation by restricting usual $\mathbb{C}^{|S|}$ differentiation to the one-dimensional fiber. Recall that we started our whole discussion with a system with a finite number $|S|$ of eigenvalues and we used S as an index set for them. Therefore, in a basis we have

$$\boldsymbol{\nabla} |\mathbf{v}\rangle = |n\rangle \langle n | \mathrm{d}\mathbf{v}\rangle \tag{45}$$

and a connection ω is given by

$$\boldsymbol{\nabla} |n\rangle = |n\rangle \langle n | \mathrm{d}n\rangle = |n\rangle \, \omega \tag{46}$$

Given a smooth curve in the base space $C : [0, 1] \to M$, the section along this curve is its lift $\tilde{C} : [0, 1] \to E$ such that $\pi \circ \tilde{C} = C$. If $\mathbf{v}$ is its tangent vector field, parallel translation means $\boldsymbol{\nabla}\mathbf{v} = 0$ along the curve. Let's calculate that:

$$\boldsymbol{\nabla}(v\, |n\rangle) = (\mathrm{d}v + v\omega)\, |n\rangle \stackrel{!}{=} 0 \tag{47}$$

Since this is a complex line bundle, $v(X) = e^{i\beta(X)}$, so that this implies

$$d\beta = i\omega = i \langle n | \mathrm{d}n\rangle \tag{48}$$

Furthermore, we note that

$$\boldsymbol{\nabla} |\mathbf{v}\rangle = |n\rangle \, e^{i\beta} e^{-i\beta} \langle n | \mathrm{d}\mathbf{v}\rangle = |\mathbf{v}\rangle \langle \mathbf{v} | \mathrm{d}\mathbf{v}\rangle = 0 \tag{49}$$

implies $\mathrm{Im} \langle \mathbf{v} | \mathrm{d}\mathbf{v}\rangle = 0$. The holonomy of a curve is now given exactly by Berry's phase. Unsurprisingly, we have recovered all the previous results!

D. A simple proof of Chern's theorem

Before showing an example of everything so far, we will prove a deep geometrical result, namely Chern's theorem. Consider a closed, oriented 2-surface V embedded in M, and pick a path $C \subseteq V$. Now parallel-transport a particle along that path. It acquires a geometric phase, γ_C. By Stokes' theorem, this can be expressed as an integral over the surface, let's call it $\Sigma^+ \subseteq V$, such that $\partial\Sigma^+ = C$. Now suppose you parallel-transported it back along the same path. This time, it picks up a phase $-\tilde{\gamma}_C$ and we express it as the integral of a 2-surface $\Sigma^- = V - \Sigma^+$. Now, this can't possibly be right unless these phases add to a multiple of $2\pi i$. This means

$$\gamma_C - \tilde{\gamma}_C = \int_{\Sigma^+} F^+ + \int_{\Sigma^-} F^- = \int_V F \in 2\pi i\mathbb{Z} \tag{50}$$

Here one integral flipped sign because its area element is oriented in the opposite direction. Here the notation suggests that there are only the two patches, $\Sigma^\pm$, but the integration may need to be done over more that just two. Thus, by requiring consistency, we have proved [7]

Theorem III.1 *(Chern) Let E be a hermitian line bundle with a purely imaginary connection. Let V be a closed, oriented $2-$surface as above and let F be the curvature. Then*

$$\frac{i}{2\pi}\int_V F$$

is an integer. The integrand is called the Chern form, or the first Chern characteristic.

Alternatively, simply contract the loop to a point. The identity loop cannot give a nontrivial phase. Note that in this approach we ought to have considered actually shrinking the loop around some Dirac string type singularity, otherwise we cannot contract the loop, so this approach is actually a bit trickier.

IV. EXAMPLES

A. Particle in a magnetic field

Consider a spin-$1/2$ particle in a slowly varying magnetic field $\mathbf{B}$. The Hamiltonian is given by

$$H(\mathbf{B}) = \frac{1}{2}\mu_B\, \boldsymbol{\sigma}\cdot\mathbf{B} \tag{51}$$

and there are two eigenvectors such that

$$H(\mathbf{B})\,|\pm(\mathbf{B})\rangle = \pm\frac{1}{2}\mu_B|\mathbf{B}|\,|\pm(\mathbf{B})\rangle \tag{52}$$

Parametrizing $\mathbf{B} = B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$ will give us the eigenvectors explicitly after a short calculation. We obtain

$$|+(\mathbf{B})\rangle = (\cos\theta/2, e^{i\phi}\sin\theta/2) \tag{53}$$

$$|-(\mathbf{B})\rangle = (-\sin\theta/2, e^{-i\phi}\cos\theta/2) \tag{54}$$

We see that they only depend on the angles, therefore our parameter space is actually S^2. We note that this representation is ill-defined at the poles. More specifically, at the north pole we have $|+\rangle = (1,0)$ but $|-\rangle = e^{-i\phi}(0,1)$. Similarly, $|+\rangle$ is multivalued at the south pole. We will, of course, have two connections, but these will also inherit this problem. We can calculate them from (21):

$$A^+ = i\,\langle +|\mathrm{d}+\rangle = -\frac{1}{2}\,(1-\cos\theta)\,\mathrm{d}\phi \tag{55}$$

$$A^- = i\,\langle -|\mathrm{d}-\rangle = +\frac{1}{2}\,(1+\cos\theta)\,\mathrm{d}\phi \tag{56}$$

We note that these differ by $-\mathrm{d}\phi$, which means they are related by a gauge transformation. They should both give the same curvature, however they are related by a parity transformation which flips orientation: $(\theta,\phi)\mapsto(\pi-\theta,\pi+\phi)$. Therefore we add extra signs $F^\pm = \pm\mathrm{d}A^\pm = \mp\frac{1}{2}\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi$, which means

$$\gamma_C = \mp\frac{1}{2}\Omega(C) \tag{57}$$

where $\Omega(C)$ is the solid angle subtended by the curve whose boundary is C. This corresponds to a magnetic monopole of strength $g = \mp 1/2$. In general, it can be shown that for any spin we have

$$\gamma_C = -m\Omega(C) \tag{58}$$

where m is the projection of the spin on the z-axis. Note the sign ambiguity! Any loop defines two solid angles since it cuts S^2 into two different surfaces. Depending on the orientation of the curve, these differ by $\pm 4\pi$, which thankfully changes γ_C up to a factor of 2π, so all is well. Note also the validity of Chern's theorem!

The Aharonov-Bohm effect is another example of geometric phases at work. Here consider an electron being dragged around a (for simplicity) infinitely thin solenoid. Even though the magnetic field vanishes everywhere outside the solenoid, the vector potential does not. Since the Hamiltonian is $U(1)$ gauge invariant, we can immediately identify the vector potential with Simon's connection. In this problem, the vector potential is given by

$$A = -i\frac{\Phi}{2\pi}\mathrm{d}\phi \tag{59}$$

where we have identified our parameter space with S^1 to consider only rotations around the solenoid. Here Φ is the total flux, kept fixed as we shrink the solenoid. Now parallel transport gives us a phase

$$\mathrm{Im}\int_C A = -\frac{\Phi}{2\pi}\phi \tag{60}$$

Note that the connection is flat, ie $F = \mathrm{d}A = 0$, and that Stokes' theorem doesn't apply because of the obstruction at the center of the punctured plane.

B. Generalizations

We will also generalize some of the results. First let's mention the case of N-fold degenerate eigenvalues. In this case, parallel transport with period T yields

$$|n_a(T)\rangle = e^{i\gamma_d} U_{ab}(C)\,|n_b(0)\rangle \tag{61}$$

where $\{|n_a\rangle\}$ belong to the same degenerate subspace. Now we need

$$U_{ab}(C) = \mathcal{P}\exp\left\{\left(i\oint_C A_{ab,\mu}\mathrm{d}x^\mu\right)\right\} \tag{62}$$

where $\mathcal{P}$ signifies path ordering, and $A_{ab,\mu} = \langle n_a|i\partial_\mu n_b\rangle$. This connection is called the Wilczek-Zee gauge potential and we are dealing with $U(N)$, a non-abelian gauge group. We may of course look at parallel transport on any principal bundle, but we will not be able to associate a physical system with unless the fibers are sums of various $U(N)$'s.

The Aharonov-Andanan (AA) phase is a nonadiabatic generalization, with loops in state space itself and not a parameter space. We consider density matrices as main objects of study, and consider a loop such that $\rho(T) = \rho(0)$. This results in a dynamical phase

$$\gamma_d = -\int_0^T dt\,\langle\psi(t)|\,H(t)\,|\psi(t)\rangle \tag{63}$$

as well as the geometric AA phase,

$$\gamma_{AA} = \oint\langle\psi(t)|i\mathrm{d}\psi(t)\rangle = \iint \mathrm{Tr}\rho\,\mathrm{d}\rho\wedge\mathrm{d}\rho \tag{64}$$

There is even a nonunitary analog discovered by Pancharatnam in which we divide an ensamble into two identical subensables and then subject one of them to a sequence of measurements (nonunitary projections). In the case that this system returns to its initial state, it again gains a dynamical and geometric phase. We refer the reader to [4] for a review.

V. CONCLUSION

We have given a brief overview of some basic geometrical aspects of Berry's phase, accompanied by a few examples and analogies. But the main point wasn't really Berry's phase, it was the importance of geometry in quantum evolution. Thus a seemingly unimportant phase factor has a lot to say about how it lives on a $U(1)$ principal bundle and measures holonomy, and the Aharonov-Andanan approach shows that this isn't even restricted to adiabatic evolution. It is common to talk about gauge transformations in QFT, yet here we encounter it in nonrelatvistic quantum mechanics. And it's measureable! Our aim to attempt to demistify why this phase exists and what's so geometrical about it has hopefully been achieved.

Appendix A: More of Berry's curvature

Let us note two additional presentations of Berry's curvature, which we label with F_n to clarify which eigenket it is associated with. We can insert a complete basis system $\sum_m |m\rangle\langle m|$ as follows:

$$F_n = -\mathrm{Im}\,\langle\mathrm{d}n|\wedge|\mathrm{d}n\rangle \tag{A1}$$

$$= -\mathrm{Im}\sum_{m\neq n}\langle\mathrm{d}n|m\rangle\wedge\langle m|\mathrm{d}n\rangle \tag{A2}$$

where the $m = n$ term drops out because it is real. Next we note that due to $\langle n|\partial_k m\rangle = -\langle\partial_k n|m\rangle$ we have the following identity

$$\sum_n F_n = -\mathrm{Im}\sum_{n,m}\langle\mathrm{d}n|m\rangle\wedge\langle m|\mathrm{d}n\rangle \tag{A3}$$

$$= -\mathrm{Im}\sum_{n,m}\langle n|\mathrm{d}m\rangle\wedge\langle\mathrm{d}m|n\rangle \tag{A4}$$

$$= +\mathrm{Im}\sum_{n,m}\langle\mathrm{d}m|n\rangle\wedge\langle n|\mathrm{d}m\rangle \tag{A5}$$

$$= -\sum_n F_n = 0 \tag{A6}$$

This was also another justification for adding a minus sign in our discussion of the two level system, but we note that by using the form (A2) we can calculate this explicitly.

Now we recall (5) which we can reformulate as

$$\langle m|\mathrm{d}n\rangle = -\frac{\langle m|\,\mathrm{d}H\,|n\rangle}{E_m - E_n}. \tag{A7}$$

We insert this into (A2) to obtain

$$F_n = -\mathrm{Im}\sum_{m\neq n}\frac{\langle n|\,\mathrm{d}H\,|m\rangle\wedge\langle m|\,\mathrm{d}H\,|n\rangle}{(E_n - E_m)^2} \tag{A8}$$

We can use this result to obtain Berry's angle for a particle of arbitrary spin j [1], because

$$dH = {}^1\!/_2\,\mu_B d(\boldsymbol{\sigma}\cdot B\mathbf{r}) = {}^1\!/_2\,\mu_B B\boldsymbol{\sigma}\cdot\mathrm{d}\mathbf{r} \tag{A9}$$

Note that accidental degeneracies ruin this approach. In the two-level system we considered, this happens only at the origin. Clearly, there is a geometrical reason obstruction to constructing line bundles at these submanifolds.

Appendix B: A quantum "metric tensor"

Since F_n is the imaginary part of something, it would stand to reason that there is a gauge invariant tensor that corresponds to this something. There isn't much choice here, we need to build it from $\langle\partial_i n|\cdots|\partial_j n\rangle$. It turns out

$$T_{ij} = \langle\partial_i n|\,(1 - |n\rangle\langle n|)\,|\partial_j n\rangle \tag{B1}$$

is such a choice. Its imaginary part is exactly $2\,(F_n)_{ij}$, and its real part is symmetric and clearly connected to a natural notion of distance in Hilbert space, $1 - |\,\langle 1|2\rangle\,|^2$. The infinitesimal version of this distance indeed yields exactly this metric.

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